Probability, A Reintroduction

1 Background and Review

Recall that a *probability*, also known as a *chance*, is a number showing how likely some event is to happen. Let us call the event X. Then the probability of X taking place,

$$P(X) = \frac{\text{The number of the outcomes such that } X \text{ happens.}}{\text{The number of all the possible outcomes.}}$$

Problem 1.1. *A standard 6-sided die has exactly six faces, each having a distinct number from 1 to 6. Imagine that you throw one die.*

- *What is the probability that you get the number 1?*
- *What is the probability that the number you get is odd?*
- *What is the probability that the number is at least 3?*

Problem 1.2. *Consider a game in which you toss two dice and sum their numbers up. List all possible outcomes of the two dice such that the sum of two dice is 2. What is the probability that you get a sum of 2?*

Problem 1.3. *Suppose you have a red die and a blue die. List all possible outcomes of the two dice such that their sum is 3. What is the probability that you get a sum of 3? What if the dice are not colored?*

Problem 1.4. *Imagine you now throw five dice at the same time.*

- *What is the probability that you get 5 as the sum of all dice? (Hint:* $6^5 = 7776$*.)*

- *What is the probability that you get 6 as the sum?*

What if we want to calculate the probability of multiple events happening? Simple! Suppose X and Y are both events. We follow a similar definition to that of a single event happening:

$$P(X \text{ and } Y) = \frac{\text{The number of the outcomes such that } X \text{ and } Y \text{ both happen.}}{\text{The number of all the possible outcomes.}}$$

$$P(X \text{ or } Y) = \frac{\text{The number of the outcomes such that either } X \text{ or } Y \text{ (or both) happen.}}{\text{The number of all the possible outcomes.}}$$

You can think of this as calculating the probability of a new event Z, which corresponds to (X and Y) or (X or Y). This extends to more than two events. Note: We call two events A and B *disjoint* or *mutually exclusive* if they never occur together:

$$P(A \text{ and } B) = \emptyset$$

Problem 1.5. *Imagine you throw a single die. What is the probability that the number rolled is both odd and at most 3?*

2 Conditional Probability

So far, you have been presented with various scenarios and asked to calculate the the probability of a certain event happening. However, as is often the case in the real world, what if we are provided with some additional information? How do we take this information into account when calculating the probability of our event happening?

Problem 2.1. *Your friend tosses two fair coins. A fair coin is equally likely to turn up heads or tails when flipped.*

- *What is the probability that both of the coins turn up heads?*

- *Suppose you snuck a peek at the first coin and saw that it was heads. What is the probability that both of the coins turn up heads?*

Formally, we say the probability of an event Y *conditioned* on a possible event X is:

$$P(Y|X) = \frac{P(X \text{ and } Y)}{P(X)}$$

Note that this formula doesn't make sense when X is impossible, as we can't divide by 0. Let's rewrite this equation to see why it holds true:

$$P(X \text{ and } Y) = P(X) \cdot P(Y|X)$$

Intuitively, the probability of two events happening is equivalent to the probability of one event happening multiplied by the probability of the other event happening *once it is known that the first event happens.*

Problem 2.2. *Is the italicized part of the previous statement necessary? Could we have instead said "the probability of two events happening is equivalent to the probability of one event happening multiplied by the probability of the other event happening?" Provide an explanation if so, and a counterexample if not.*

Example 1. *Imagine you roll two dice. Suppose A = (Sum of the two dice rolls is 6) and B = (First roll is 4). We would have:*

$$\begin{aligned} P(A|B) &= \frac{P(A \text{ and } B)}{P(B)} \\ &= \frac{P(\textit{Sum of the two rolls is 6 and first roll is 4})}{P(\textit{First roll is 4})} \\ &= \frac{P(\textit{Second roll is 2 and first roll is 4})}{P(\textit{First roll is 4})} \\ &= \frac{\frac{1}{6} \cdot \frac{1}{6}}{\frac{1}{6}} \\ &= \frac{1}{6} \end{aligned}$$

Problem 2.3. *During 2021 in Alberta, 61.9% of the population were fully vaccinated. 87.7% of COVID-19 hospitalizations were unvaccinated or partially vaccinated. There are 8,124 total hospitalizations of a total population of 4.371 million. What is the probability of being hospitalized if you are vaccinated versus the probability of being hospitalized if you are unvaccinated or partially vaccinated? (Math 170E, Spring 2024)*

Problem 2.4. *In a city of 2 million people, 20% are under 20 years old. If there are 5,000 car accidents annually and 60% involve drivers under 20, what is the probability of having an accident if you are under 20 compared to being 20 or older?*

Problem 2.5. *Of a company's 10,000 employees, 80% have health insurance. If there were 1,000 visits to the hospital and 75% were by insured employees, what is the probability of visiting the hospital if you are insured versus uninsured?*

Problem 2.6. *Your friend tosses two coins, and this time they are careful to hide the outcomes from your prying eyes. However, they let you choose to look at one of the coins, and you see that it came up heads. What is the probability that both of the coins are heads?*

3 Independent Events

We say that events A and B are *independent* if $P(A \text{ and } B) = P(A)P(B)$.

Problem 3.1. *Using the equation of conditional probability, $P(X \text{ and } Y) = P(X) \cdot P(Y|X)$, prove that if two events, A and B, are independent, then $P(A|B) = P(A)$, and $P(B|A) = P(B)$. In other words, prove that events A and B do not influence the probabilities of each other.*

Problem 3.2. *You flip a fair coin twice. What is the probability that both tosses are heads? Are these events independent?*

Problem 3.3. *From a standard deck of 52 cards, you draw one card and then another without replacing the first. Are these draws independent? What is the probability of drawing two aces?*

Problem 3.4. *If you roll two six-sided dice, what is the probability that both dice show a number greater than 4? Consider whether the outcome of one die affects the other.*

Problem 3.5. *Can two disjoint events, A and B, also be independent? If yes, provide a condition where this holds.*

What if there are three events? We say events A, B and C are *mutually independent* if:

1. A, B and C are pairwise independent, meaning each pair of events is independent.
2. $P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C)$

Problem 3.6. *Show that the second condition above is necessary. That is, construct events A, B and C such that they are all pairwise independent, but $P(A \text{ and } B \text{ and } C) \neq P(A) \cdot P(B) \cdot P(C)$*

What if we have even more events? We say a set of events $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ is mutually independent if:

1. Every proper subset of events $S \subsetneq \mathcal{A}$ is mutually independent.
2. $P(A_1 \text{ and } A_2 \text{ and } \cdots \text{ and } A_n) = P(A_1) \cdot P(A_2) \cdots P(A_n)$

Problem 3.7. *Construct n events such that any set of fewer than n events is mutually independent, but the set of all n events is not mutually independent.*

4 Expectation

A *random variable* is a collection of outcomes, each with some assigned value and some assigned probability. For example, a fair dice roll can be thought of as a random variable X which can take on values between 1 and 6, inclusive. Furthermore, each of these dice rolls has equal probability:

$$P(X = 1) = P(X = 2) \cdots = P(X = 6) = \frac{1}{6}$$

Note: you can think of a random variable taking on a value ($P(X = 2)$) or a set of values ($P(X \leq 4)$) as satisfying our earlier definition of an event.

The *expected value* of a random variable X with possible outcomes $k_1, k_2, \ldots, k_n$ and respective probabilities $p_1, p_2, \ldots, p_n$ is:

$$E[X] = k_1 \cdot p_1 + k_2 \cdot p_2 + \cdots + k_n \cdot p_n$$

Expected value of a random variable can be thought of as the value of the random variable averaged over many trials.

Example 2. *Let there be a game where you enter by paying \$2, and throw a fair dice. Your earning is the dollar amount of the outcome of the dice. Calculate your expected profit and indicate whether you should enter the game or not.*

Let $X = \$1, \$2, ..., \$6$ *be the possible rewards based on the outcome of dice, where each reward has a probability of 1/6. Then, we would calculate the expected value as:*

$$E[X] = \$1 \cdot (1/6) + \$2 \cdot (1/6) + \cdots + \$6 \cdot (1/6) = 21/6 = \$3.5$$

We paid \$2 dollars to enter this game, and our expected earning is \$3.5. Therefore, our expected profit is \$3.5 - \$2 = \$1.5, which indicates that we should enter this game.

Problem 4.1. *Expectation is an important topic in the fair price in chance games. Suppose a slot machine pays out \$1 one-fourth of the time, \$2 one-fifth of the time, and a jackpot of \$100 with a probability of 1/1000. Calculate the "fair" price to enter this game.*

Problem 4.2. *There are 30 students in a Math Circle session that attempt to solve a math problem. Suppose half of the students answer it correctly with a probability of 1/2, and the other half answer it correctly with a probability of 3/5. One answer is chosen at random at let X be equal to 1 if answer is correct, and 0 otherwise. Calculate E[X].*

Problem 4.3. *In a particular roulette game, a player can bet \$1 on a single number. If the player wins, they receive \$36 (including their original bet). The probability of winning (the ball landing on the chosen number) is 1/37 as there are 37 numbers on the wheel. Calculate the expected value for a player's bet and determine whether the game is fair.*

5 Financial Literacy Chapter: Return and Expected Return of Financial Assets

In finance, an **asset** is a resource with economic value that an individual, corporation, or country owns or controls with the expectation that it will provide a future benefit. Often used interchangeably, an **investment** is an asset with the goal of appreciation, i.e. an increase of the value of the asset, or generating income, i.e. providing regular cash payments as we see in rental income. To label the change in the value of an asset, we use the term **return**. In simplest terms, return is the money made or lost on an investment over some time. (Investopedia)

A popular measure of return is the Holding Period Return, abbreviated as HPR, which gives an investor's return over time that is owned by a particular investor. The holding period can be expressed nominally or as a percentage (Investopedia):

$$HPR = \frac{\text{Ending Investment Value + Income - Beginning Investment Value}}{\text{Beginning Investment Value}}$$

Example: Say you buy a house with $100,000. You then let your apartment for 1 year, earning $10,000 of rental income in total. At the end of year 1, you sell your apartment for $120,000. Calculate your HPR.

$$HPR_{house} = \frac{120,000 + 10,000 - 100,000}{100,000} = 30\%$$

Investing in Stocks: A **stock** is a financial asset that represents the fractional ownership of the issuing corporation. Units of stocks are called **"shares"**, and buying a **share** of a company means buying a fraction of the ownership of a company. Stocks are bought and sold in stock markets and are the foundation of many investors' portfolios.

Problem 5.1. *Suppose you think that Apple is a promising company and its value will appreciate over time. Therefore, you buy a share of Apple for \$100, wait for 3 years, and sell the share for \$250. Calculate your HPR.*

Problem 5.2. *You buy gold for \$1,200 per ounce and hold onto it for 3 years. Over this period, the value of gold appreciates to \$1,500 per ounce. There are no additional earnings from the gold during the holding period. Calculate the HPR of this gold investment.*

Expected Return in Finance: The concept of expectation has significant applications in finance. For example, as stocks are bought and sold on a stock exchange market, such as NASDAQ or NYSE, a large number of analysts and professionals in the financial world try to analyze and predict the past and present movements of these prices, adjusting their investments accordingly to the meet their investment return demands and make as much profit within reasonable risk levels. Therefore, such analysts often provide estimations on how a particular stock or the entire stock market may act in the future with different probabilities.

Problem 5.3. *Suppose you want to buy an Apple share currently trading at \$95 per share. You found this stock price estimation for Apple stock from the report of an investment bank based on three different scenarios regarding the stock market's trajectory in the next 3 months. What is your expected HPR? Use the previous formula for the expected value and combine it with the formula of HPR.*

State of Stock Market	Probability	Forecast of Stock Price
Boom (Optimistic)	0.35	150
Normal Growth	0.30	100
Recession (Decline)	0.35	90

Table 1: Apple Stock Price Forecast

6 Goat, Goat, Car

Problem 6.1 (Monty Hall Problem)**.** *You are on a game show, being asked to choose between three doors. Behind each door, there is either a car or a goat. You choose a door. The host, Monty Hall, picks one of the other doors, which he knows has a goat behind it, and opens it, showing you the goat. (You know, by the rules of the game, that Monty will always reveal a goat.) Monty then asks whether you would like to switch your choice of door to the other remaining door. Assuming you prefer having a car more than having a goat, do you choose to switch or not to switch?.*