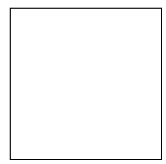
Surfaces

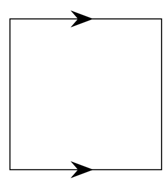
Prepared on May 10, 2024 by Max

Part 1: Fundamental Polygons

Imagine a square sitting in the Euclidean plane. We use "square" here to mean the solid square, not just the four lines (just like a disk versus a circle).



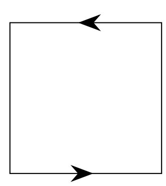
Now, we glue the top side and the bottom side (without a twist).



We draw these arrow heads on the top and the bottom to denote which direction they are glued together (in this case, the same direction, which is to say without a twist).

Problem 1:
What surface does this make?
We haven't defined what "surface" means. For now, use your intuition for what a surface is. We will define "surface" later.

Problem 2:
What surface do we get if we glue the top and bottom in the opposite directions?

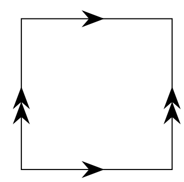


Problem 3:
Take a square of paper[1] and determine all the surfaces you can make by identifying one edge with another edge (with or without a twist). How many different surfaces can you make?

Of these surfaces, one is completely impossible to actually make with a piece of paper.
In the next step, we will discuss more surfaces that are impossible to actually make. Our goal today will be to understand these surfaces and why they can't be made.

[1] For some of the surfaces it may be easier to use a rectangle (e.g. by cutting/ripping a thin strip of paper off of the side of this packet).

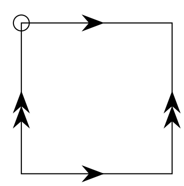
Consider a new square of paper. Now, we glue the top side and the bottom side (without a twist), and then glue the left side and the right side (without a twist).



We use double-headed arrows to signify that the left and right sides are glued, not, say, the left and bottom sides.

Problem 4:
What surface does this make? Does it matter what order we do the gluing in?
If you've ever played Pac-Man, you might know about screen-wrapping. When you go off the side of the board in one direction, you appear on the other side. In effect, Pac-Man is played on this surface!

We call a square with any number of indicated gluings a **fundamental square** for a surface. We can easily describe fundamental squares in the following way (called the **generator** of the fundamental square):



- Starting at the top-left corner, we go around the square clockwise
- When we pass over an arrow, we add a number equal to the number of heads it has (so our first edge, the top edge, we add a 1)
- If the arrow is going the opposite direction to the way we travel, we mark the number with an inverse (e.g. 1^{-1})
- If an edge has no arrows, we add a 0

So for this square, our generator is $1\,2^{-1}\,1^{-1}\,2$.

Problem 5:
Are generators unique to a surface? That is, given a surface that comes from a fundamental square, is there a unique generator for that surface?

Problem 6:
Find a generator for the Möbius Strip.

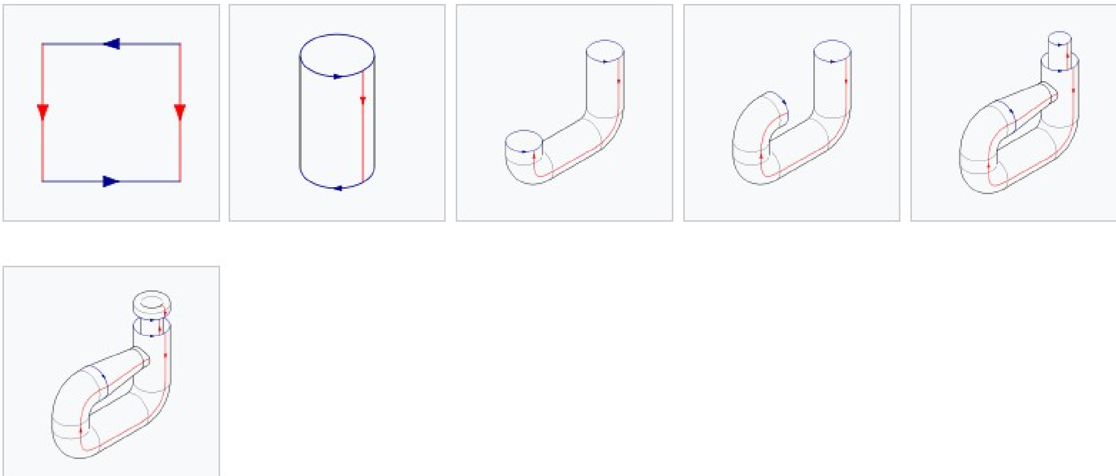
Whenever we encounter objects that are "finite" in nature (there are only so many ways to pair edges of a square), a natural question is if they can be classified.

Problem 7:
Is there a fundamental square with generator $1\,1^{-1}\,2^{-1}\,2$? What surface does it produce?

Problem 8:
Draw every possible fundamental square. Which give the same surfaces? Which surfaces are possible to actually make with a piece of paper?

Part 2: Embeddings

Let's look at a famous example of a surface that cannot be *embedded* into three-dimensional space, the Klein Bottle. (We will define what "embedding" means shortly.)



(Wikipedia) – how the Klein bottle is constructed.

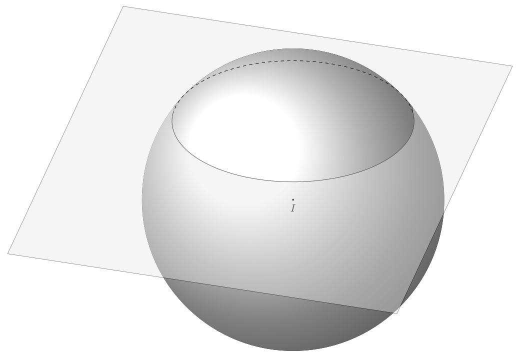
In order to understand the Klein bottle, we need to formally define what a surface is.

Definition 9:
A **surface embedded in** d **dimensions** is a set X that can be viewed as a subset of $\mathbb{R}^d$ so that for every element $x \in X$, there exists a d-dimensional ball S centred at x so that $S \cap X$ looks like a disk or half-disk[2].

Informally, we say X is **embedded** in d dimensions to mean that it fits into d dimensions. A line can't fit into 0 dimensions, but it can be embedded in 1 dimension, or 2 dimensions, or 3, et cetera. A (usual 3-dimensional) sphere can't fit into 2 dimensions, but it can be embedded in 3 dimensions, or 4, etc.
To understand this definition, let's work through an example.

[2] The phrases "looks like" and "can be viewed as" are not precise. They can be made formal using topology, but we don't have the background to do that.

Example 10:
Let's show that a sheet of paper is a surface embedded in 3 dimensions. Take a piece of paper (maybe this one!) and choose your favourite point. Imagine a very very small sphere in three dimensions centred at that point. If the point you chose was on the edge of the paper, the intersection of the sphere and the paper will be a half-disk. If your point was not on the edge and the sphere was small enough, the intersection will be a full disk.



However, just because a piece of paper can be embedded in 3 dimensions doesn't mean it can only be embedded in 3 dimensions.

Problem 11:
Show that a piece of paper is also a surface embedded in 2 dimensions.

Problem 12:
If X is a surface embedded in d dimensions, show that it is also a surface embedded in $d+1$ dimensions. *Hint: what is the intersection of a $d+1$-dimensional sphere and a d-dimensional sphere? That is hard to visualise, so think about $d = 2$ and try to generalise from there.*

By induction, we see that if a surface X is embedded in d dimensions and $m > d$, then X is also embedded in m dimensions. Thus, we can define the **embedding dimension** of X to be the smallest positive integer d so that X can be embedded in d dimensions.

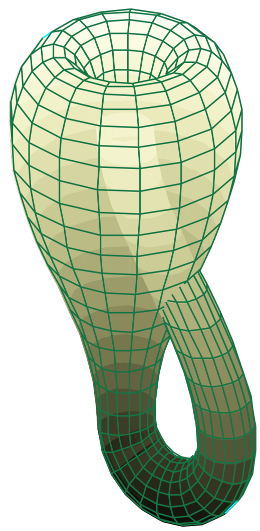
Problem 13:
Prove that the Möbius Strip is a surface embedded in 3 dimensions.

Problem 14:
Can a Möbius Strip be embedded in 2 dimensions? Why or why not? *Hint: remember that the Möbius Strip has only one side – it has no outside and no inside. Can this happen in two dimensions?*

Problem 15:
Find the embedding dimension of a Möbius Strip.

It's very hard to think about dimensions greater than 3. However, a lot of the very interesting surfaces we want to consider can't be embedded in 3 dimensions!

Problem 16:
Prove that the Klein bottle cannot be embedded in 3 dimensions. *Hint: consider a point on the intersection between the stem of the bottle and the rest of the bottle, and use the definition of an embedded surface.*



(Wikipedia)

It turns out that Klein bottle can be embedded in 4 dimensions and *immersed* (a word we won't define) in 3 dimensions. The figure above is an immersion but not an embedding of the Klein bottle in $\mathbb{R}^3$. If we treat colour as a "fourth dimension," then this picture actually demonstrates an embedding of the Klein bottle into $\mathbb{R}^4$!

Given a surface $X \subset \mathbb{R}^d$ embedded in d dimensions, we define the **boundary** of X to be the subset $Y \subset X$ of all points that satisfy the "half-disk" part of the definition of an embedded surface. We denote the boundary of X by ∂X.
Intuitively, for a piece of paper embedded in 3 dimensions, the boundary is all four edges.

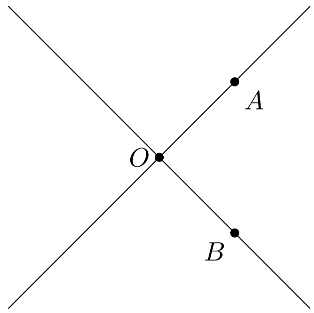
Problem 17:
Find the boundary of the Möbius Strip. *Hint: consider the fundamental square.*

Problem 18:
Find the boundary of the Klein bottle.

Part 3: Projective Plane

The *projective plane* is the surface with generator 1 2 1 2. You can't make this with a piece of paper, but it's still a super interesting surface. We will construct it in two new ways (other than the fundamental square). Explaining why they are the same is left as an exercise.

Consider two lines through the origin in the Euclidean plane $\mathbb{R}^2$. If we choose an arbitrary point on the line, this uniquely determines the line, since two points determine a line.



But what about the converse? Given two points in $\mathbb{R}^2$, how can we determine if they represent the same line through the origin?
Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$, with $A \neq (0,0)$ and $B \neq (0,0)$. Let ℓ_A be the line through A and $(0,0)$, and ℓ_B be the line through B and $(0,0)$.

Problem 19:
Determine when $\ell_A = \ell_B$.

Now, let's move to three dimensions. We can extend the previous problem to determine when two points in $\mathbb{R}^3$ represent the same line through the origin.

We can define a set $X := (\mathbb{R}^3 \setminus \{(0,0,0)\})/\sim$ where two points $x, y \neq 0 \in \mathbb{R}^3$ satisfy $x \sim y$ iff $\ell_x = \ell_y$. The notation $\mathbb{R}^3/\sim$ means that for every point $x \in \mathbb{R}^3$, we find the set of all points $\{y \mid x \sim y\}$, which we denote $[x]$ (called the **class** of x). Clearly $x \in [x]$ since $x \sim x$, and we define $\mathbb{R}^3/\sim := \{[x] \mid x \in \mathbb{R}^3\}$.

In fact, we have $[x] = \left[\frac{x}{|x|}\right]$ when $x \neq 0$.

Problem 20:
Why is this true?

Then since $\left|\frac{x}{|x|}\right| = \frac{|x|}{|x|} = 1$, we see that for every $x \neq 0 \in \mathbb{R}^3$, there is some $y \in S^2 \subset \mathbb{R}^3$ in $\mathbb{R}^3$ with $[x] = [y]$ (S^2 is the unit sphere in $\mathbb{R}^3$). Thus, $X = S^2/\sim$.

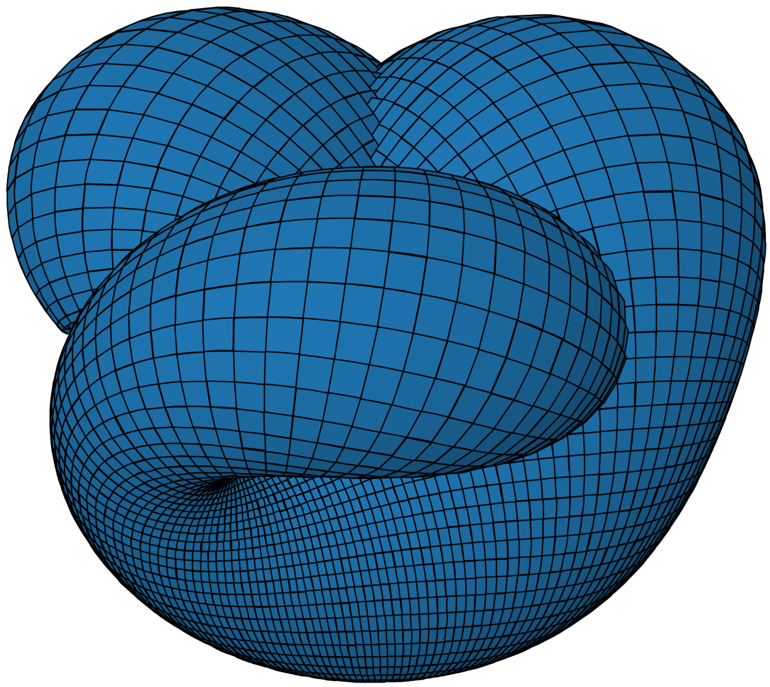
Problem 21:
Why is $X = S^2/\sim$?

But for any $x, y \in S^2$, we have $x \sim y$ iff $x = \pm y$. We say x and y are **antipodal** if $x = -y$, so we can say $x \sim y$ if and only if $x = y$ or x and y are antipodal.

Problem 22:
Why do we have $x \sim y$ iff $x = \pm y$?

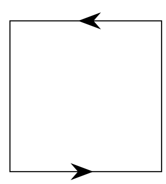
Thus, X is the unit sphere S^2 where we glue every pair of antipodal points together. This is exceptionally difficult to visualise! But in the next step we will present an entirely different presentation of this space that is more visual.

Let's describe the projective plane explicitly. Just like the Klein bottle, it cannot be embedded in three dimensions.



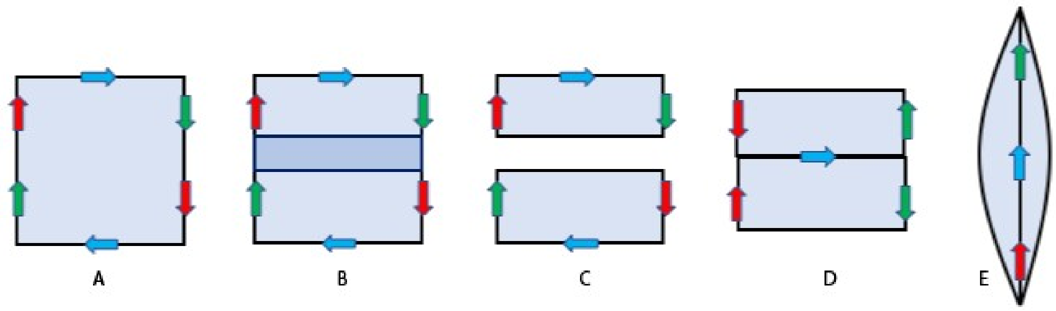
(Wikipedia)

This surface is known as Boy's Surface[3], and is the best possible way of fitting the projective plane into three dimensions. It is not an embedding because it has self-intersections, but it has a "minimal amount" of self-intersection (ask an instructor if you curious about what this means).
To make it, we start with a Möbius Strip.



In Problem 17, we discovered that the boundary of the Möbius Strip is a circle. If we glue a disk onto the boundary of the Möbius Strip, we get Boy's Surface.
We can also construct it in a (rather silly, in my opinion) way via the fundamental square.

[3]Named after Werner Boy who discovered it.

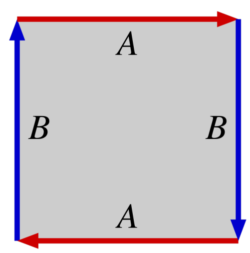


(Michael Laszlo)

A: The fundamental square of the projective plane
B: The shaded region in the centre is a Möbius Strip
C: We remove the Möbius Strip
D: We join the top and bottom edge
E: We are left with a disk

So working backwards, we have a construction of the projective plane from a disk and a Möbius Strip, glued along their boundaries.

Problem 23:
Explain why these two constructions represent the same surface, and are the same as the fundamental square with generator 1 2 1 2.



(Wikipedia)

Part 4: Extremely Challenging Problems

Problem 24:
Given a fundamental square for a surface X, prove that $\pi_1(X) = \langle 1, 2 | g \rangle$ where g is the generator of the fundamental square. (Note: as written, this may be slightly wrong depending on how exactly you construct the generator. It's up to you to figure out the details.)

Problem 25:
Prove that every (compact, aka bounded) surface embedded in d dimensions has a fundamental polygon (not necessarily a square).

It's actually true that every compact surface is either a sphere or a "connected sum" of copies of the torus and the projective plane. The Klein bottle is $\mathbb{P}^1 \# \mathbb{P}^1$, where $\#$ is the connected sum.

Problem 26:
Prove the *Whitney Embedding Theorem*: the embedding number of a surface is at most 4.