Tilings

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1 What are tilings?

Given some planar region, and a number of tiles of some shape (maybe distinct for each tile), a tiling is a configuration of the tiles on the plane that covers the entire region without any overlaps. Note that for the purpose of this worksheet the tiles can be reflected and rotated in any way. For example:

Using the following region and the given tiles:

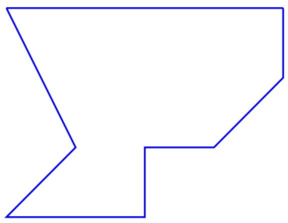


Figure 1: Region

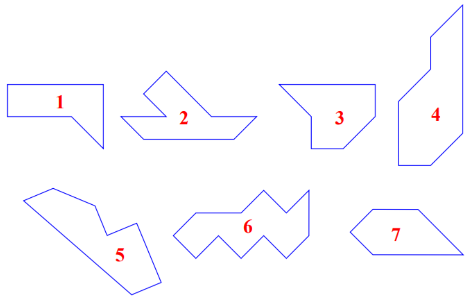
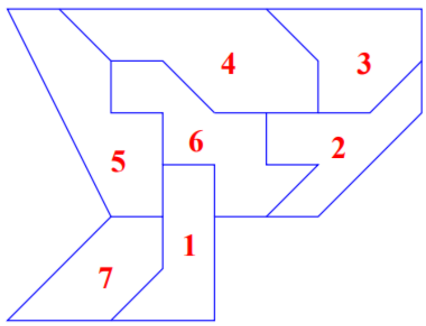


Figure 2: Tiles

An acceptable tiling is the following:

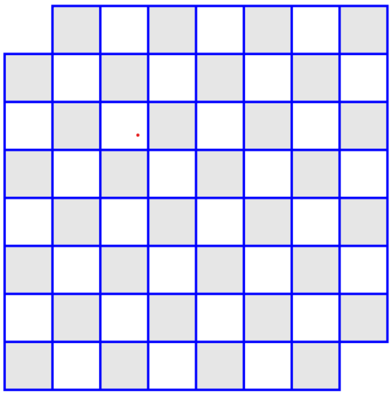


In this worksheet we will not focus so much on finding specific tilings but on discovering whether they are possible or not. In other words, given a region and some tiles, **is it possible to completely cover that region with the given tiles without any overlaps?**

2 Tilings and Colorings

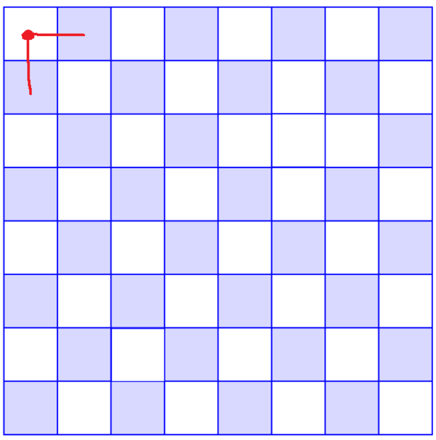
Problem 2.1.
Show that if f_n is the number of ways you can tile an $n \times 2$ grid using 2×1 dominoes, then $f_n = f_{n-1} + f_{n-2}$.

Problem 2.2.
An 8×8 chessboard has the bottom right and top left corners cut out. Can it by tiled by 2×1 dominoes?



Given a grid, we will define a **rectangular path** to be a path that starts on a square of the grid and can move only horizontally or vertically one grid square at a time (through edges).

Problem 2.3.
On an 8×8 chessboard draw a rectangular path starting and ending on the top left grid square that traverses all the squares once each.



Problem 2.4.
We will define the length of a rectangular path to be the number of squares it traverses, including the starting square. Starting at B1 which squares can you reach with a rectangular path of even length?

Problem 2.5.
Remove a black and a white square from the chessboard. Looking at the rectangular path you drew before, by removing these squares you have divided this path into two shorter ones. Do these paths have even or odd length (not including the missing squares)?

Problem 2.6.
Show that if we remove a black and a white cell from the chessboard, we can tile the remaining board using 2×1 dominoes.

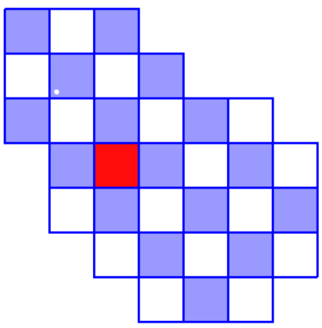
Problem 2.7.
Can you find a rectangular path that traverses all the squares of a chessboard with the two black corners missing.

Problem 2.8.
A piece of cheese has a shape of a $3 \times 3 \times 3$ cube with the central piece missing. A mouse starts eating a corner cube, and after finishing a cube moves to one of the adjacent cubes. Can the mouse eat all the cheese?

Generally, when our grid is not rectangular, we have the following useful theorem:

Theorem 1.
A region cannot be tiled by 2×1 dominoes if we one can find k cells of one color with fewer than k neighbors (adjacent cells).

Problem 2.9.
Show that it is impossible to tile the following using 2 by 1 dominoes. With red we denote a missing cell.

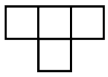


2.1 More Colorings

A tetromino consists of 4 total 1×1 cells that are connected in some configuration along their edges.

Problem 2.10.
Two teterominoes are considered identical if they are rotations or reflections of each other. How many unique tetrominoes are there?

e.g. the T-tetromino:



Problem 2.11.
Show that the 10×10 board cannot be tiled with T tetrominoes.

Problem 2.12.
Think of a different black and white coloring to show that it is impossible to tile a 10×10 board with L - shaped tetrominoes.

Problem 2.13.
Prove the same results for T and L tetrominoes on $(4n + 2) \times (4n + 2)$ sized boards.

Problem 2.14.
Prove that you can tile a $4n \times 4n$ board with T and L tetrominoes.

Hard (Optional):

Problem 2.15.
Think of a different coloring to show that you cannot tile a 8×8 board with a corner removed using 3×1 trominoes.

Problem 2.16.
Think of yet another clever coloring to show that you cannot tile a 10 by 10 board with 1×4 tetrominoes.

Problem 2.17.
Ninety nine 2×2 squares were cut out of a 29×29 board. Prove that it is always possible to cut at least one more 2×2 square.

3 Integer Grids

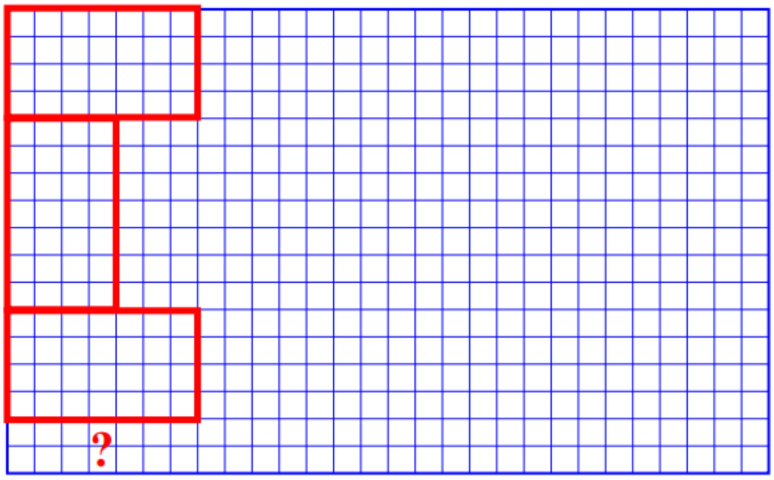


Figure 3: A 24×17 grid being tiled using 7×4 rectangles

Let us now consider a different family of problems: **tiling an $m \times n$ rectangular grid using $a \times b$ sized rectangles** when m, n, a, and b are all natural numbers. One obvious restriction is that $mn/ab = \textit{number of tiles used}$, which has to be an integer. In other words, ab has to divide mn.

Another constraint can be seen by looking at the outermost columns and rows of the $m \times n$ grid. (left and right column, and top and bottom row)

Problem 3.1.
If I can tile an $m \times n$ grid with $a \times b$ tiles. Then:

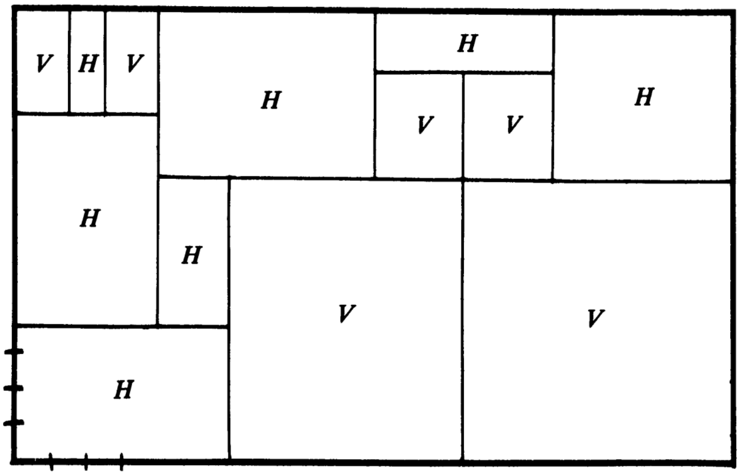
- $m = k_m a + l_m b$ for some positive integers k_m, l_m
- $n = k_n a + l_n b$ for some positive integers k_n, l_n

However, this condition still doesn't guarantee that such a tiling can be found. For example, it is impossible to tile a 10×15 grid using 1×6 rectangles.
To find the final condition we shall prove a more general result.

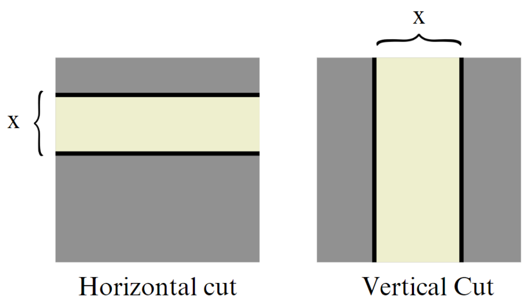
Theorem 2.
Whenever a rectangle is tiled by rectangles each of which has at least one integer side, then the tiled rectangle has at least an integer side.

In other words, our rectangle tiles can now vary in shape but the only condition we have is that either their horizontal side length, or their vertical side length is an integer.

This means that we could split the tiles into two categories H-tiles and V-tiles, i.e. tiles with a horizontal integer side, and ones with vertical integer sides. An example of such a tiling would be the following.



Problem 3.2.
Given a 1×1 square find a coloring using the same amounts of black and white, such that any vertical cut or horizontal cut of the square, of size $(x \times 1)$ and $(1 \times x)$ respectively $(x < 1)$, is neutral in color, i.e. contains an equal amount of black and white. *Hint: You don't have to try anything too complicated.*



Problem 3.3.
Fill an m by n grid by coloring each 1×1 with the coloring you found above. Show that if a rectangle has at least one integer side then regardless of where you place it it will always cover the same amount of black and white.

This means that if we color our $m \times n$ grid in this manner then our H-tiles and V-tiles are neutral in color. So, if they tile the entire grid then the grid has to be neutral in color as well.

Problem 3.4.
Show that a rectangle with its bottom left corner aligned with the corner of the grid, that has two non-integer sides cannot contain the same amount of black and white.

Problem 3.5.
Conclude that at either m or n has to be an integer if this tiling is possible.

Theorem 3.
If we can tile an $m \times n$ grid using $a \times b$ rectangle tiles, where m, n, a, and b are natural numbers then:

- a divides either m or n
- b divides either m or n

Problem 3.6.
Use Theorem 2 and a re-scaling argument to prove the above theorem.

This conditions actually don't only show us a tiling could be possible, but in fact that it definitely is.

Problem 3.7.
If a divides m and b divides n, then can you see how a simple tiling of an $m \times n$ grid using $a \times b$ rectangles would look like.

Problem 3.8.
If a divides m and b divides m, then can you see how a simple tiling of an $m \times n$ grid using $a \times b$ rectangles would look like.