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A Study of Measurement Errors as an Introduction to Differential Calculus

Harder problems are marked with a red pepper sign.

Let $\bar{x}$ be a quantity we'd like to measure and let x be the result of the measurement. The *absolute error* of the measurement, $\Delta(x)$, is the smallest positive real number we can find such that

$$|\bar{x} - x| \leq \Delta(x). \tag{1}$$

The triangle inside (1) is the upper-case Greek letter *Delta*. Using a slightly more extensive wording, we say that the number $\bar{x}$ we don't know is no further from the number x we do know than $\Delta(x)$.

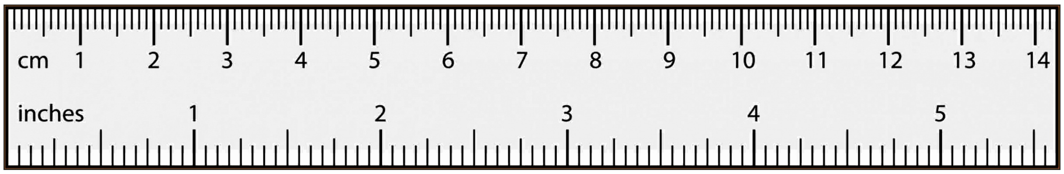
$$x - \Delta(x) \leq \bar{x} \leq x + \Delta(x)$$

Question 1 *How tall are you?*

Problem 1 *You need to choose a letter to express your height. Would you use x or $\bar{x}$? Explain your choice.*

Problem 2 *Alice's height is $x = 5'8''$. What do you think $\Delta(x)$ is? Why?*

Let x be a measurement. Then its precision cannot be better than a half of the smallest unit of the measuring device.



For example, if you measure the length with a standard metric ruler, you cannot be more precise than a half of a millimeter. If you measure some length with a ruler similar to the above and write down the result as $x = 56.734$ *mm*, it just does not make sense! The best you can do in the case $\bar{x} = 56.734$ *mm* is to round to the nearest mark, $x = 57$ *mm*. If $\bar{x} = 56.449$ *mm*, then $x = 56$ *mm*. The case $\bar{x} = 56.5$ *mm* is the hardest because you can round either way, $x = 56$ *mm* or $x = 57$ *mm*. Therefore, the absolute error for measuring with the above ruler is $\Delta(x) = 0.5$ *mm*.

This brings about the idea of *significant digits* a.k.a. *significant figures*. The idea is that the last significant digit shows the precision the number bears. For example, if $x = 34.07$ and all the digits are significant, then $\Delta(x) = 0.005$.

Problem 3 *All the digits of the following numbers are significant. Find the absolute errors.*

- $x = 25$ $\Delta(x) =$
- $x = 0.007$ $\Delta(x) =$
- $x = 1000$ $\Delta(x) =$
- $x = 1000.0001$ $\Delta(x) =$
- $x = 2.5 \times 10^7$ $\Delta(x) =$
- $x = 2.5 \times 10^{-7}$ $\Delta(x) =$
- $x = 0$ $\Delta(x) =$
- $x = 0.000$ $\Delta(x) =$

Problem 4 *There are $x = 3$ cars parked in front of a house.*

- $\bar{x} =$ $\Delta(x) =$
- *How many digits in the above number $x = 3$ are significant?*

Problem 5 *Given* $x = 25$, $\Delta(x) = 0.1$ *and* $y = 15$, $\Delta(y) = 0.2$, *find* $\Delta(x + y)$.

Problem 6 *Given* $x = 25$, $\Delta(x) = 0.1$ *and* $y = 15$, $\Delta(y) = 0.2$, *find* $\Delta(x - y)$.

Problem 7 *Prove the following formula, called the sum/difference rule:*

$$\Delta(x \pm y) = \Delta(x) + \Delta(y) \tag{2}$$

Problem 8 *Given* $x = 2$, $\Delta(x) = 0.1$ *and* $y = 5$, $\Delta(y) = 0.2$, *find* $\Delta(x \times y)$.

Assume for simplicity that $x > 0$ and $y > 0$. Further assume that $\Delta(x)$ is much smaller than x and $\Delta(y)$ is much smaller than y, whatever this means. Then

$$x - \Delta(x) \le \bar{x} \le x + \Delta(x)$$
$$y - \Delta(y) \le \bar{y} \le y + \Delta(y)$$

implies

$$-x\Delta(y) - y\Delta(x) + \Delta(x)\Delta(y) \le \bar{x}\bar{y} - xy \le x\Delta(y) + y\Delta(x) + \Delta(x)\Delta(y)$$

Taking the engineering approach, let us drop the product $\Delta(x)\Delta(y)$ as too small compared to any other term. Then

$$|\bar{x}\bar{y} - xy| \le x\Delta(y) + y\Delta(x)$$

Comparing this to (1) establishes the following formula, known as the *product rule*:

$$\Delta(xy) = x\Delta(y) + y\Delta(x) \tag{3}$$

Problem 9 *All the digits of all the numbers involved are significant. Find the result and its absolute error.* $(2.5 \times 10^2 - 1.23 \times 10^1) \times (8.00 \times 10^{-1}) =$

The following is the standard definition of the limit of a sequence: for $\{a_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} a_n = A \iff \forall \epsilon > 0,\ \exists N \in \mathbb{N} : n \ge N \Rightarrow |A - a_n| < \epsilon$.

Problem 10 *Prove that* $\lim_{n\to\infty} q^n = 0$ *for any* $q \in \mathbb{R}$ *such that* $|q| < 1$.

Problem 11 *Prove that for* $q \ne 1$, $1 + q + q^2 + q^3 + \ldots + q^n = \dfrac{1 - q^{n+1}}{1 - q}$.

Problem 12 *Prove that for* $|q| < 1$, $\sum_{n=0}^{\infty} q^n = \dfrac{1}{1-q}$.

If q is small, whatever this means, the formula derived in Problem 12 shows that

$$1/(1-q) \approx 1 + q. \tag{4}$$

Problem 13 *Use (4) to show that*

$$\Delta\left(\frac{1}{y}\right) = \frac{\Delta(y)}{y^2}$$

Problem 14 *Use the formula from Problem 13 to prove the following formula, called the* quotient rule.

$$\Delta\left(\frac{x}{y}\right) = \frac{x\Delta(y) + y\Delta(x)}{y^2} \tag{5}$$

Problem 15 *All the digits of all the numbers involved are significant. Find the result and its absolute error.* $(2.5 \times 10^2 - 1.23 \times 10^1) \div (5.00 \times 10^{-1}) =$

Problem 16 *Use (3) and mathematical induction to prove the following formula, known as the* power rule:

$$\Delta(x^n) = nx^{n-1}\Delta(x). \tag{6}$$

Problem 17 *Use (6) to prove that*

$$\Delta(\sqrt[n]{x}) = \frac{1}{n}x^{\frac{1-n}{n}}\Delta(x). \tag{7}$$

The following sequence of steps enables one to measure the radius of a solid sphere more dense than water.

Step 1. Take a measuring cup of known volume and fill it with water up to the rim.

Step 2. Place the sphere in the measuring cup. Some water will be displaced out of the cup. The volume of the displaced water equals that of the sphere.

Step 3. Find the volume of the sphere and the absolute error of the measurement.

Step 4. Use the formula

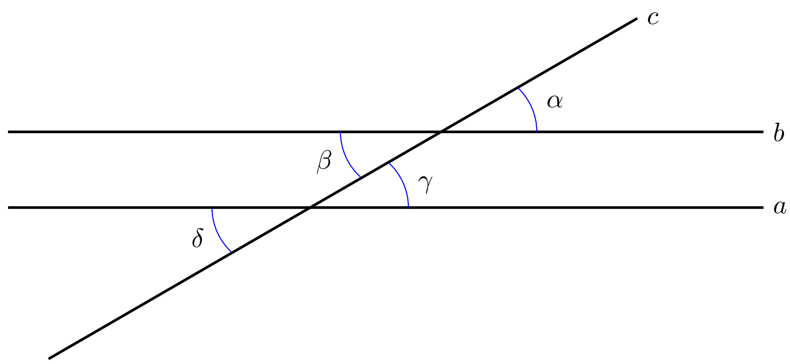
$$V = \frac{4}{3}\pi r^3$$

to find the radius of the sphere.

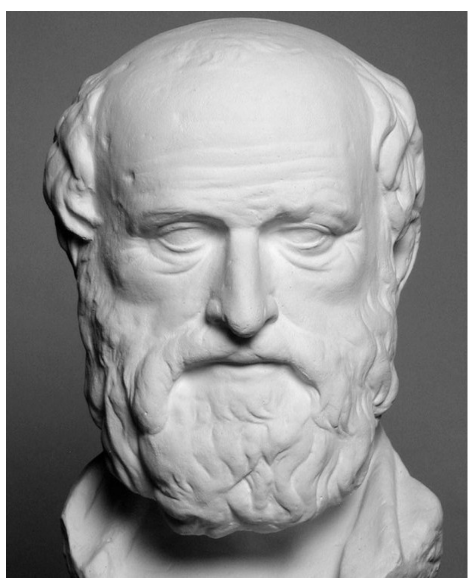
Problem 18 *Find the absolute error of measuring the radius. Note: don't forget that* π *also comes with a rounding error.*

How to measure the radius of the Earth with a camel and a stick

Proposition 1 *If straight lines a and b are parallel and a straight line c intersect them as shown on the picture below, then the angles α, β, γ, and δ are congruent.*

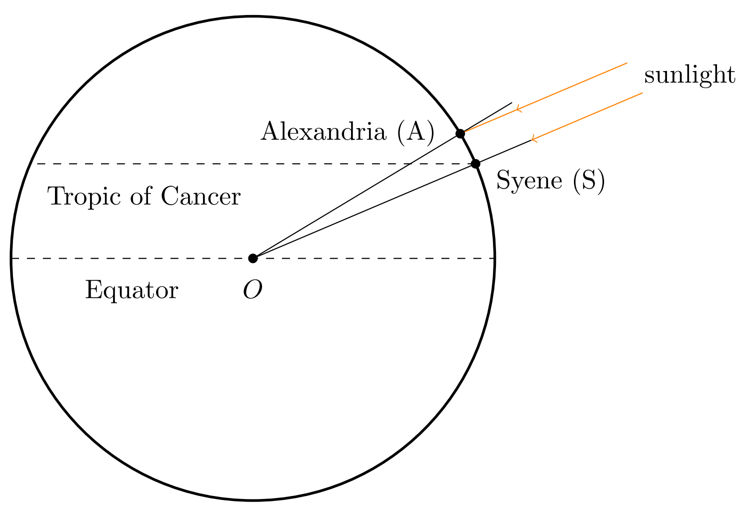


Eratosthenes, the first man to measure the Earth, was born in 276 BC in Cyrene, a polis founded by the Greeks in 630 BC, now a city in the country of Libya. Appointed the chief librarian of the Great Library of Alexandria, the most famous research and teaching institution of the ancient world, the distinguished mathematician, astronomer and geographer was also renowned as a music theorist, poet, and athlete.



The average distance from the Earth to the Sun, also known as the astronomical unit (au), is 149,597,870,700 m or about 150,000,000 km. The average radius of the Earth, $\bar{R} = 6,371$ km, is the microscopic 0.004% of the au. For practical purposes of Earthly geometry, one can think that the Sun is infinitely far away and that, because of this, all the rays of light coming from the Sun to the Earth are parallel.

The ancient city of Syene, currently Aswan in Egypt, is located on the Tropic of Cancer, $23°26'12.7''$ North of the Equator, exactly south of Alexandria, $31°12'56.3''$ North of the Equator. If we stick a pole vertically in the ground, so that its lower end points to the center of the Earth, in Syene, the shadow the pole casts will disappear at noon of the Summer solstice day. Since Alexandria lies North of the tropic, the shadow of a similar vertically stuck pole will not disappear at this moment of time. The shadow and the pole will form the legs of a right triangle giving one a chance to measure the angle opposite to the ground. Thanks to Proposition 1 well known to Eratosthenes, the angle will be equal to the angle between the pole and the sunlight as well as to the angle AOS between the Alexandria and Syene poles.



The below formula (8) and problem 19 will help us reverse-engineer the measurement carried out by Eratosthenes.

For a small angle α measured in radians,

$$\sin\alpha \approx \tan\alpha \approx \alpha. \tag{8}$$

We are not going to prove (8) in this mini-course. The proof is standard for any calculus course. Instead, please check the validity of the formula by making a few experiments with a calculator. Make sure you use radians, not degrees!

Problem 19 *Assume that the length p of the pole Eratosthenes used was two meters. Recall that Alexandria and Syene are nearly on the same meridian and that their latitudes are $31°12'56.3''$ N and $23°26'12.7''$ N respectively. Find s, the length of the shade cast by the Alexandria pole.*

Let us take the value of s found in problem 19 as a result of Eratosthenes's measurement. Let us call α the angle between the Alexandria pole and the sunlight. Then

$$\tan\alpha = \frac{s}{p} \approx \alpha. \tag{9}$$

Problem 20 *Assuming $\Delta(p) = \Delta(s) = 1$ cm, find $\Delta(\alpha)$.*

Eratosthenes knew that the length l of a circumference was related to its radius R via the formula $l = 2\pi R$.

Once we know the angle α between the rays OA and OS, all we need to estimate the radius R of the Earth is the length l of the arc AS. Indeed,

$$\frac{l}{2\pi R} = \frac{\alpha}{2\pi}. \tag{10}$$

The story has it that Eratosthenes took a camelback journey from Alexandria to Syene, counting the camel's steps on the way. To figure out l, he multiplied the number of the steps by the average step length. Since 10 is equivalent to the equation

$$R = \frac{l}{\alpha},$$

he proceeded to calculate the radius of our planet.

Problem 21 *Use Google Earth's ruler to estimate the distance l between Alexandria and Aswan. Make a practical assumption with regard to $\Delta(l)$.*

Problem 22 *Compute the Earth's radius R and $\Delta(R)$. Compare to the known average radius of the Earth, $\bar{R} = 6,371$ km.*

Question 2 *Taking a longer look at the line connecting the modern day Aswan to Alexandria, one begins to wonder how the camel-back journey up North was possible at all. It is not easy to cross Nile twice riding a camel.*

Did the Nile's riverbed move West over the years?

Relative error

Question 3 *You measure your own height, h, and the radius of the Earth, R, with the same absolute error, $\Delta(h) = \Delta(R)$. Which measurement is more precise?*

Question 4 *When you are two and your brother is twelve, the absolute age difference is the same as when you are 50 and he is 60. However, the age difference seems to get much less with age. What's going on?*

The *relative error*,

$$\delta(x) = \frac{\Delta(x)}{|x|} \times 100\%, \tag{11}$$

is often a better measure of the precision of a measurement. The Greek letter δ is the lower-case *delta*. The notations (Δ and δ) are terrible, but, alas, quite common.

Problem 23 *Prove that* $\delta(xy) = \delta(x) + \delta(y)$.

Problem 24 *Solve the functional equation* $f(xy) = f(x) + f(y)$.

Question 5 *What kind of function is* δ*?*