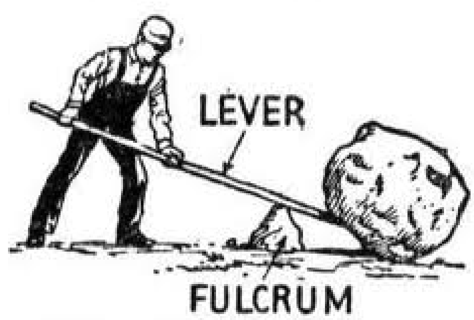
Baricentric coordinates

Weighted sums are closely related to the workings of the first machine invented by the humans, a *lever*. The first man to understand the workings of the machine was Archimedes of Syracuse, 287-212 BC.

A lever consists of a solid beam rotating around a fixed point, a *fulcrum* or *pivot*. The force applied to one side of the lever results in the force being exerted at the opposite side.

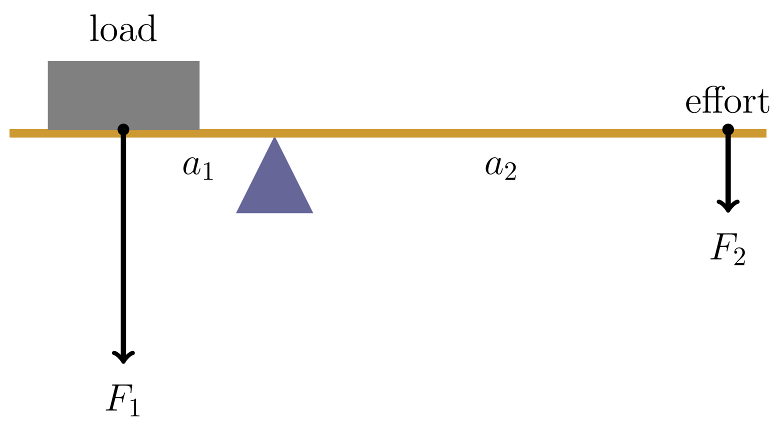


Man lifting a stone with a lever

The distance a from the fulcrum to the point where a force F is applied is called the *arm* of the force. The product

$$T = aF \tag{1}$$

is called the *moment of force* or *torque*. The arm on the load side, a_1 on the picture at the top of the next page, is called the *resistance arm*, the arm on the opposite side, a_2 on the picture below, is called the *effort arm*.



The lever is in balance when the torque of the load equals to that of the effort.

$$a_1 F_1 = a_2 F_2 \tag{2}$$

The load starts moving when the torque of the effort exceeds the torque of the load.

Problem 1 *The curb weight*[1] *of a popular Toyota Sienna minivan is 4,000 lbs. A person weighing 200 lbs wants to lift the car with a lever. The resistance arm of the lever is 3'. How long should be the effort arm?*

Problem 2 *A boy weighing 50 lbs wants to lift his 200 lbs father using a 5-foot-long stick as a lever. Where should he place the fulcrum? Hint: a picture will help.*

[1] The total weight of a vehicle with standard equipment and all the necessary liquids, including a full tank of fuel, motor oil, coolant, etc. while not loaded with either passengers or cargo.

Let x be a point on the number line in between zero and one.

$$0 > x > 1$$

Consider the segment $[0, 1]$ as a lever. The weights w_0 and w_1 one should place at zero and one respectively so that the lever with the fulcrum at x would be in balance are called the *barycentric coordinates* of x.

Problem 3

a. Let $x = 0.25$ *and* $w_0 = 3$. *Find* w_1.

b. Find w_1 *for the same* x *and* $w_0 = 12$.

c. Find w_1 *for the same* x *and* $w_0 = 30$.

d. Find w_1 *for the same* x *and* $w_0 = 60$.

e. Does the ratio $w_0 : w_1$ *in parts a – d change? Why or why not?*

Problem 3 shows that barycentric coordinates $(w_0 : w_1)$ of a point are unique up to a common non-zero factor. Coordinates of this kind are called *projective coordinates*. Barycentric coordinates are a special, and very important, type of projective coordinates.

Problem 4 *Find barycentric coordinates of the following points. Draw pictures if needed.*

- $1/2$

- $1/7$

- 0.8

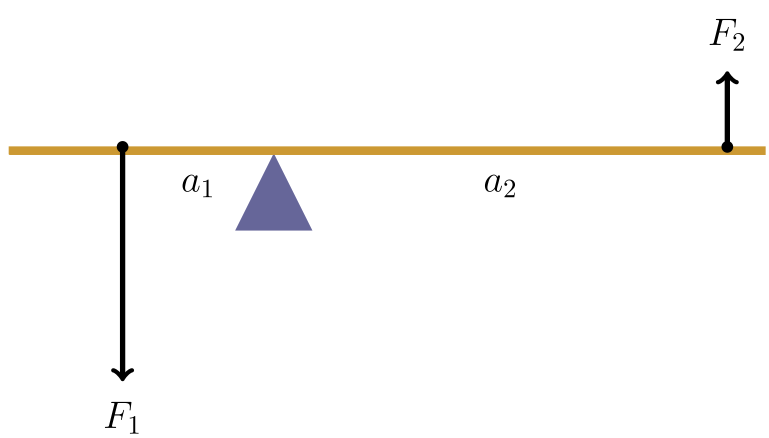
- 0.99

- 0

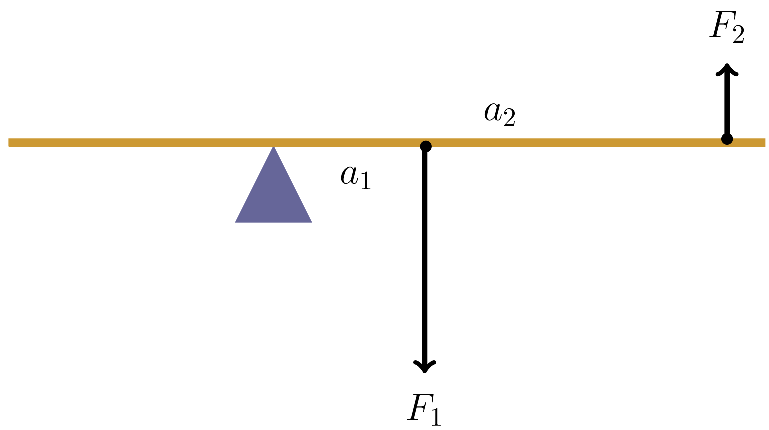
- 1

Question 1 *Can we extend barycentric coordinates to the entire number line? How?*

The idea is to use negative weights. If a positive weight pushes the lever down, then a negative weight pushes it up! However, if we place two weights, one positive and one negative, at the opposite sides of the fulcrum, their torques will rotate the rod in the same direction. We will get not a see-saw, but a merry-go-round!

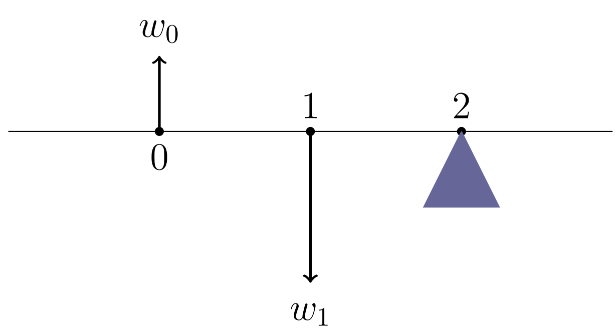


Placing two weights of opposite signs at the same side of the fulcrum brings the lever back to normal mode of operation.



Example 1 *Find barycentric coordinates of the point 2.*

Since the points 0 and 1 are on one side of the lever, we must use weights of opposite signs.



To have the lever with the fulcrum at 2 in balance, we need the weights to satisfy the following equation.

$$2w_0 + w_1 = 0$$

The weights $w_0 = -1$ and $w_1 = 2$ satisfy the formula above, so barycentric coordinates of the point 2 are $(-1 : 2)$.

Note that any pair of real numbers with the same ratio will do. For example, $(3 : (-6))$ are also barycentric coordinates of the same point.

Problem 5 *Find barycentric coordinates of the following points. Draw pictures if needed.*

- 3
- 4
- 100
- -5

- -12

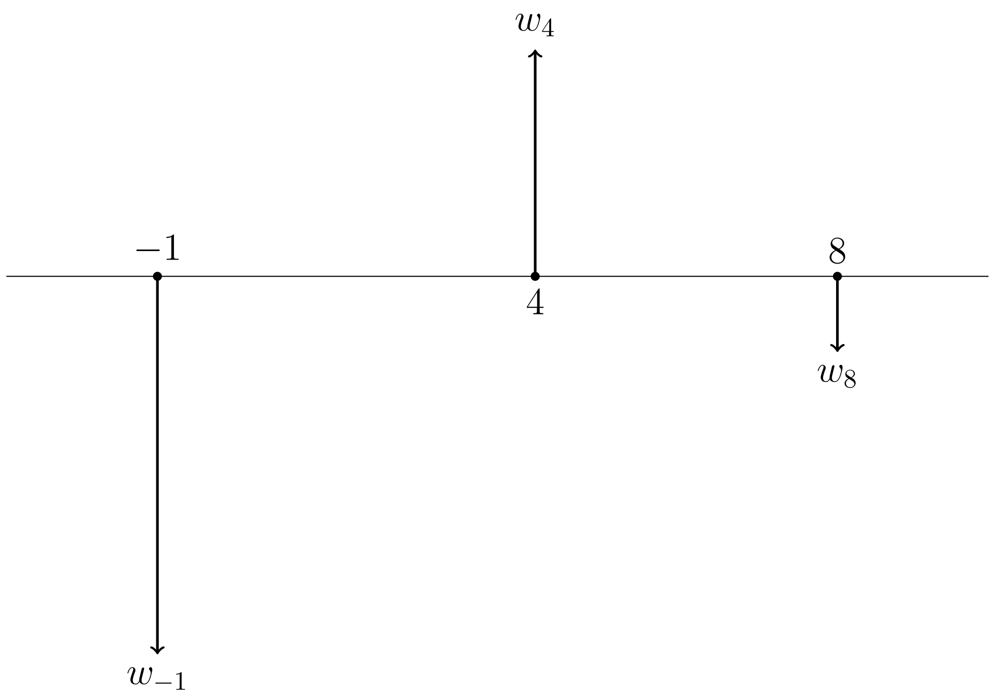
- 2.25

We have chosen the points zero and one as the reference points. However, this choice is not essential, any two different points on the number line will do.

Problem 6 *What weights, w_{-5} and w_7 should we place at the points -5 and 7 to have a lever with the fulcrum at -4 in balance?*

Note that we can have more than two forces acting on a lever. Just as above, the lever will be in balance when the sum of the torques on one side equals to the sum of torques on the other.

Example 2 *Where should we place the fulcrum to have the lever with the weights $w_{-1} = 5$, $w_4 = -3$ and $w_8 = 1$ (located at the points -1, 2, and 8 respectively) in balance?*



If the fulcrum is placed at x, then the arm lengths of the forces w_{-1}, w_4 and w_8 are $x-(-1)$, $x-4$ and $x-8$ respectively. The lever will be in balance if the torques satisfy the following equation.

$$5(x+1) - 3(x-4) + 1(x-8) = 0$$

Solving the equation gives

$$x = -3.$$

The point -3 is called the *center of mass* of the above system. If we hang a weightless straight line equipped with the above weights on a thread at the point, it will not rotate. If hanged at any other point, it will.

Problem 7

- *Find the center of mass of the weights* w_4 *and* w_8 *from Example 2.*

- *Move both weights* w_4 *and* w_8 *to their center of mass. Find the center of mass of the weight* $w_4 + w_8$ *placed at the point and of the weight* w_{-1}*.*

- *Find the center of mass of the weights* w_{-1} *and* w_8 *from Example 2.*

- *Move both weights* w_{-1} *and* w_8 *to their center of mass. Find the center of mass of the weight* $w_{-1} + w_8$ *placed at the point and of the weight* w_4*.*

- *Compare the answers for the second and fourth part of this problem to that of Example 2. Do you always get the same point? Why?*

Fact 1 *To figure out the center of mass of the system of point-weights, one can take the following steps.*

1. Choose a subsystem of the system and find its center of mass.

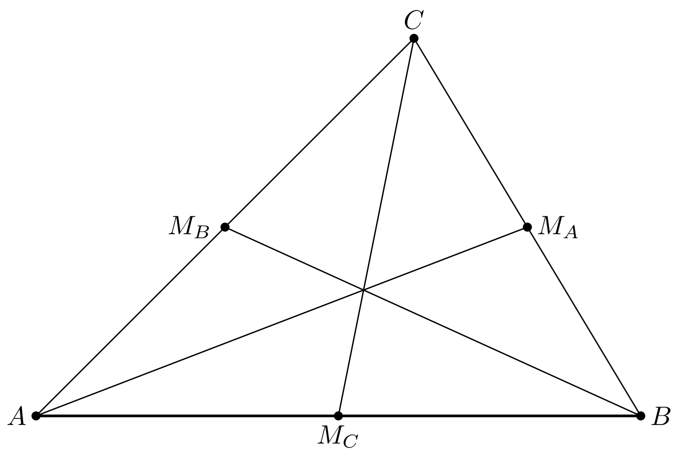
2. Move all the weights of the subsystem to its center of mass and add them up.

3. Find the center of mass of the new system.

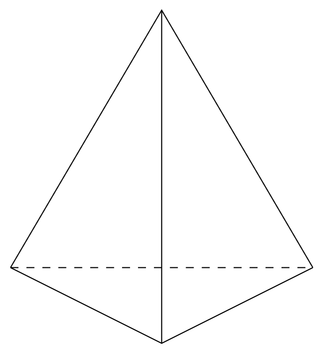
The final result is independent of the choice of the subsystem.

Problem 7 is an example of the above statement at work. We are not going to give it a proof (yet).

Problem 8 *Prove that medians of a triangle intersect at one point. The point divides each median in the ratio* $2:1$ *counting from the corresponding vertex.*



A *tetrahedron*, also known as a *triangular pyramid*, is the simplest possible solid in the Euclidean 3D.



Each face of a tetrahedron is a triangle. Imagine that we find the intersection point of the medians for every face of a tetrahedron and connect the point to the opposite vertex.

Problem 9 *Prove that all the four resulting lines intersect at one point. Prove that the point divides each of the lines in the ratio 3:1 counting from the corresponding vertex.*

Problem 10 *Draw a 4-dimensional triangular pyramid, also known as a pentachoron.*

Problem 11 *Generalize the statements of Problems 8 and 9 to four dimensions. Prove the statement.*