Intermediate 1 **April 28, 2024**

Continued Fractions

1 Finite Continued Fractions

Definition 1. *A* ***finite continued fraction*** *is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}}$$

where we will assume always that $a_1, \dots, a_n$ are positive integers. We often use the more compact notation $[a_0, a_1, \dots, a_n]$.

To compute the continued fraction expansion of a number, for example $\frac{11}{3}$, we use the following recursive procedure:

- Write $\frac{11}{3} = 3 + \frac{2}{3}$, separating out the largest whole number possible.
- Rewrite this as $3 + \frac{1}{\frac{3}{2}}$, expressing the non-whole number part $\frac{2}{3}$ as 1 over its reciprocal.
- Repeat the procedure on the denominator $\frac{3}{2}$.

If there is nothing left over after we take away the largest whole number, the procedure terminates. Here is the procedure carried out on $\frac{11}{3}$:

$$\begin{aligned}\frac{11}{3} &= 3 + \frac{2}{3} \\ &= 3 + \frac{1}{\frac{3}{2}} \\ &= 3 + \frac{1}{1 + \frac{1}{2}}\end{aligned}$$

Problem 1.1. *Compute a continued fraction expansion for each of the following numbers:*

1. $\frac{5}{12}$

2. $\frac{5}{3}$

3. $\frac{33}{23}$

4. $\frac{37}{31}$

Problem 1.2. *For each continued fraction, write the corresponding number as a reduced fraction:*

1. $[2, 3, 2]$

2. $[1, 4, 6, 4]$

3. $[2, 3, 2, 3]$

4. $[9, 12, 21, 2]$

2 Infinite Continued Fractions

The procedure outlined in the previous section may be carried out for any number x: At each step we subtract the largest whole number that we can, keeping the result non-negative. Then we invert the remainder and repeat. For example, if $x = \sqrt{2} \approx 1.414\ldots$, we have:

$$\begin{aligned}
\sqrt{2} &= 1 + \frac{1}{1+\sqrt{2}} \\
&= 1 + \frac{1}{1 + \left(1 + \frac{1}{1+\sqrt{2}}\right)} \\
&= 1 + \frac{1}{2 + \frac{1}{1+\sqrt{2}}} \\
&= \cdots
\end{aligned}$$

Problem 2.1. *Find continued fraction expansions for the square root of each natural number:*

1. $\sqrt{3}$

2. $\sqrt{4}$

3. $\sqrt{5}$

4. $\sqrt{6}$

5. $\cdots$

3 Continued Fractions, Continued

Definition 2. *Given a continued fraction*

$$[a_0, a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

*, the numbers a_j are called the **partial quotients** of the expansion. The fraction $\frac{p_k}{q_k} = [a_0, a_1, \dots, a_k]$ is called the* **k*th convergent***.

Problem 3.1. *Notice that in all the cases we have observed so far, the continued fraction expansion of a rational number has terminated.*

1. *Compute the continued fraction expansion for $\frac{195}{154}$ using the worksheet provided. Do you see a pattern in the remainders r_j?*

2. *In general, what is the relationship between a fraction $\frac{a}{b}$ and the first remainder? The first remainder and the second?*

3. *Prove that the continued fraction expansion of any rational number terminates.*

Definition 3. *A **quadratic irrational** is an irrational number which is the root of a quadratic equation $ax^2 + bx + c = 0$.*

Problem 3.2. *The "golden ratio", $\phi = \frac{1+\sqrt{5}}{2}$, satisfies $\phi^2 = \phi + 1$, hence $\phi = 1 + \frac{1}{\phi}$. What happens when you repeatedly plug the right hand equation into itself?*

Problem 3.3. *The number* $1+\sqrt{2}$ *is sometimes jokingly called the "silver ratio". Can you guess why?*

Problem 3.4. *We say that a continued fraction expansion is* ***eventually periodic*** *if the partial quotients* a_j *eventually fall into a repeating cycle. We use the notation* $[a_0, a_1, \ldots, a_k, \overline{a_k+1, \ldots, a_n}]$, *e.g.* $[1, \overline{2,3}] = [1, 2, 3, 2, 3, 2, 3, 2, 3, \ldots]$. *Find the number associated with each following continued fraction expansion:*

1. $[3, \overline{3, 6}]$

2. $[3, \overline{1, 2}]$

3. $[0, 3, \overline{2}]$

4. $[0, 1, \overline{10, 5}]$

Problem 3.5. *Prove that for any eventually periodic continued fraction expansion, there is a quadratic irrational with that expansion.*

Problem 3.6. *Many people know the decimal expansion of π to several places. However, the continued fraction expansion is much less familiar!*

1. *Use a calculator to find the first few (at least three) terms of the continued fraction expansion for π. What happens if you truncate the expansion after two terms? three terms?*

2. *Compare: If you know the first three terms of the continued fraction expansion, how many decimal places of accuracy does this give?*

Problem 3.7. *Given two fractions $\frac{a}{b}$, $\frac{c}{d}$, the fraction $\frac{a+c}{b+d}$ is called their* ***mediant****. What happens if you begin with the fraction $\frac{p_{k-2}}{qk-2}$, and repeatedly take mediants with $\frac{p_{k-1}}{q_{k-1}}$? Try this out with some of the expansions you've computed.*

Problem 3.8. *Prove that the convergents satisfy the recurrence relations*

$$p_k = a_k p_{k-1} + p_{k-2}$$

$$q_k = a_k q_{k-1} + q_{k-2}$$

(Hint: Assume the relation holds for ***any*** *continued fraction up to the kth level.)*

Problem 3.9. *Use the previous exercise to prove the identity*

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^k$$

What does this tell you about $\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}$?

j	$x_j = a_j + r_j$	a_j	$x_{j+1} = \frac{1}{r_j}$
0			
1			
2			
3			
4			
5			
6			
7			
8			

j	$x_j = a_j + r_j$	a_j	$x_{j+1} = \frac{1}{r_j}$
0			
1			
2			
3			
4			
5			
6			
7			
8			

j	$x_j = a_j + r_j$	a_j	$x_{j+1} = \frac{1}{r_j}$
0			
1			
2			
3			
4			
5			
6			
7			
8			

j	$x_j = a_j + r_j$	a_j	$x_{j+1} = \frac{1}{r_j}$
0			
1			
2			
3			
4			
5			
6			
7			
8			

j	$x_j = a_j + r_j$	a_j	$x_{j+1} = \frac{1}{r_j}$
0			
1			
2			
3			
4			
5			
6			
7			
8			

j	$x_j = a_j + r_j$	a_j	$x_{j+1} = \frac{1}{r_j}$
0			
1			
2			
3			
4			
5			
6			
7			
8			

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec

p_j	q_j	dec