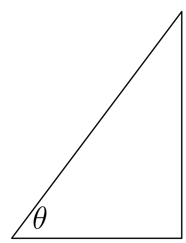
ORMC AMC 10/12 Group
Week 5: Trigonometry

April 28, 2024

1 Warm-Ups

1. Show that for any right triangle with an angle $\theta < 90°$, $\boxed{\sin^2(\theta) + \cos^2(\theta) = 1}$.



2. Given that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)},$$

   Show that $\boxed{\tan^2(\theta) + 1 = sec^2(\theta)}$ and $\boxed{1 + \cot^2(\theta) = \csc^2(\theta)}$.

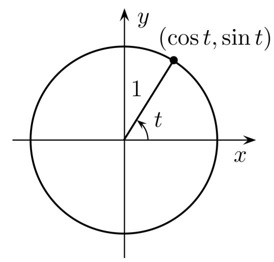
3. Calculate

$$\arcsin\left(\frac{1}{3}\right) + \arccos\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{3}\right) + \text{arccot}\left(\frac{1}{3}\right)$$

2 Facts/Theorems

Previously we have mostly considered the trigonometric functions sin, cos, tan only as they relate to triangles, specifically right triangles, where the angles in question satisfy $0° < \theta < 90°$, or $0 < \theta < \frac{\pi}{2}$ in radians.

We want to extend the definitions of these functions so that they can take any real number as arguments. In order to do so, we first consider the unit circle diagram below:



We consider the angle t (or, often θ) to be the counterclockwise distance along the unit circle that our point has moved away from the positive x-axis. And, as suggested in the diagram, the **cosine** of this angle **is the x- coordinate of the point**, and the **sine** of the angle **is the y-coordinate of the point**.

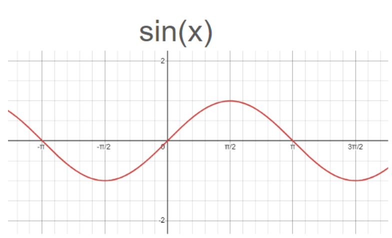
As the point moves **counterclockwise**, we consider this to be a **positive change in the angle t**. When the point moves **clockwise**, we consider this to be a **negative change in the angle t**.

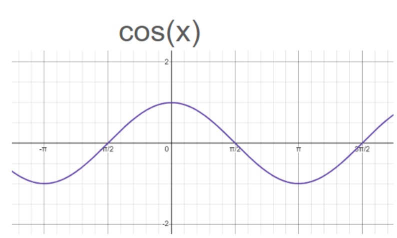
Notice that the coordinates of the point repeat once it makes a full circle. This corresponds to the angle changing by 2π radians; so $\sin(t)$ *and* $\cos(t)$ *are* ***periodic*** *with period* 2π.

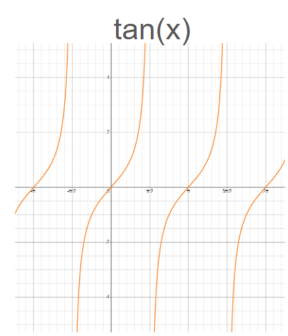
The other trigonometric functions are defined exactly the same as before, in terms of sine and cosine:

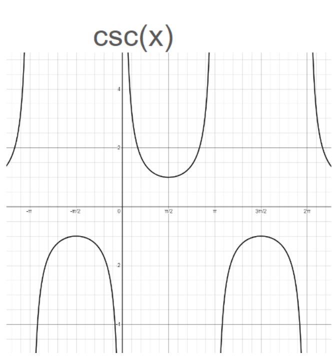
$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)}$$
$$\sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

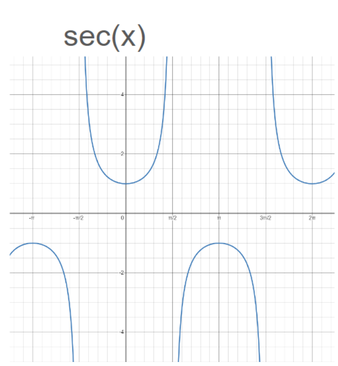
It is also important to know the graphs of these functions, which are below:

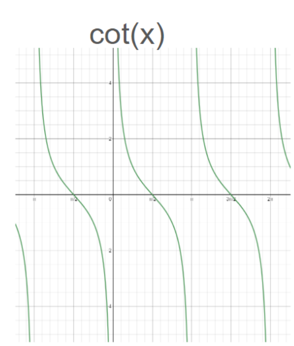












3 Exercises

1. Recall the identity $e^{ix} = \cos(x) + i\sin(x)$ which we discussed when covering complex numbers. Use this identity to show that $\boxed{\sin(2x) = 2\sin(x)\cos(x)}$ and $\boxed{cos(2x) = \cos^2 x - \sin^2 x}$.

2. Use $e^{ix} = \cos(x) + i\sin(x)$ to show that

$$\boxed{\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)}$$
$$\boxed{\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)}$$

3. Show that

$$\boxed{\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}}$$

4. Use the result of Exercise 2 to show that

$$\boxed{\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}}$$
$$\boxed{\cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2}}$$
$$\boxed{\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}}$$

5. Use the result of Exercise 4 to show that

$$\boxed{\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}$$
$$\boxed{\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}$$

6. **2017 AMC 12B #7** The functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π. What is the least period of the function $\cos(\sin(x))$?

7. **(2020 AMC 12A #9)** How many solutions does the equation $\tan(2x) = \cos(\frac{x}{2})$ have on the interval $[0, 2\pi]$?

8. **(2003 AIME I #4)** Given that $\log_{10}\sin x+\log_{10}\cos x = -1$ and that $\log_{10}(\sin x+\cos x) = \frac{1}{2}(\log_{10} n-1)$, find n.

9. **(1998 AIME #5)** Given that $A_k = \frac{k(k-1)}{2}\cos\frac{k(k-1)\pi}{2}$, find $|A_{19} + A_{20} + \cdots + A_{98}|$.

10. **(1988 AHSME #13)** If $\sin(x) = 3\cos(x)$ then what is $\sin(x)\cdot\cos(x)$?

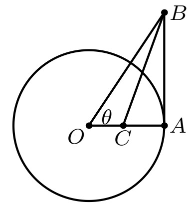
11. **(1973 AHSME #17)** If θ is an acute angle and $\sin\frac{1}{2}\theta = \sqrt{\frac{x-1}{2x}}$, then find $\tan\theta$ (in terms of x)

12. **(1989 AHSME #14)** $\cot 10 + \tan 5 =$

    (A) $\csc 5$ (B) $\csc 10$ (C) $\sec 5$ (D) $\sec 10$ (E) $\sin 15$

13. **(1999 AHSME #15)** Let x be a real number such that $\sec x - \tan x = 2$. Find $\sec x + \tan x$.

14. **(2000 AMC 12 #17)** A circle centered at O has radius 1 and contains the point A. The segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on $\overline{OA}$ and $\overline{BC}$ bisects $\angle ABO$, then $OC =$



(A) $\sec^2\theta - \tan\theta$ (B) $\dfrac{1}{2}$ (C) $\dfrac{\cos^2\theta}{1+\sin\theta}$ (D) $\dfrac{1}{1+\sin\theta}$ (E) $\dfrac{\sin\theta}{\cos^2\theta}$

15. **(2007 AMC 12A #17)** Suppose that $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$. What is $\cos(a-b)$?

16. **(2022 AMC 12A #17)** Suppose a is a real number such that the equation

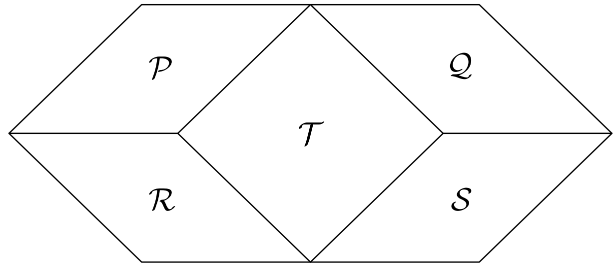
$$a \cdot (\sin x + \sin(2x)) = \sin(3x)$$

has more than one solution in the interval $(0, \pi)$. The set of all such a that can be written in the form

$$(p, q) \cup (q, r),$$

where $p, q,$ and r are real numbers with $p < q < r$. What is $p + q + r$?

17. **(2006 AIME I #8)** Hexagon $ABCDEF$ is divided into five rhombuses, $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$, and $\mathcal{T}$, as shown. Rhombuses $\mathcal{P}, \mathcal{Q}, \mathcal{R}$, and $\mathcal{S}$ are congruent, and each has area $\sqrt{2006}$. Let K be the area of rhombus $\mathcal{T}$. Given that K is a positive integer, find the number of possible values for K.



18. **(1963 AHSME #34)** In $\triangle ABC$, side $a = \sqrt{3}$, side $b = \sqrt{3}$, and side $c > 3$. Let x be the largest number such that the magnitude, in degrees, of the angle opposite side c exceeds x. Then x equals:

(A) 150° **(B)** 120° **(C)** 105° **(D)** 90° **(E)** 60°

19. **(2003 AMC 12B #21)** An object moves 8 cm in a straight line from A to B, turns at an angle α, measured in radians and chosen at random from the interval $(0, \pi)$, and moves 5 cm in a straight line to C. What is the probability that $AC < 7$?

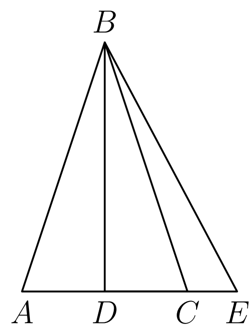
20. **(2004 AMC 12A #21)** If $\sum_{n=0}^{\infty} \cos^{2n}\theta = 5$, what is the value of $\cos 2\theta$?

21. **(1999 AHSME#27)** In triangle ABC, $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$. Then $\angle C$ in degrees is

22. **(1998 AHSME #28)** In triangle ABC, angle C is a right angle and $CB > CA$. Point D is located on $\overline{BC}$ so that angle CAD is twice angle DAB. If $AC/AD = 2/3$, then $CD/BD = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

23. **(2003 AMC 12B #23)** The number of x-intercepts on the graph of $y = \sin(1/x)$ in the interval $(0.0001, 0.001)$ is closest to

24. **(2004 AMC 12B #24)** In $\triangle ABC$, $AB = BC$, and $\overline{BD}$ is an altitude. Point E is on the extension of $\overline{AC}$ such that $BE = 10$. The values of $\tan \angle CBE$, $\tan \angle DBE$, and $\tan \angle ABE$ form a geometric progression, and the values of $\cot \angle DBE$, $\cot \angle CBE$, $\cot \angle DBC$ form an arithmetic progression. What is the area of $\triangle ABC$?



25. **(2007 AMC 12A #24)** For each integer $n > 1$, let $F(n)$ be the number of solutions to the equation $\sin x = \sin(nx)$ on the interval $[0, \pi]$. What is $\sum_{n=2}^{2007} F(n)$?

(A) 2014524 (B) 2015028 (C) 2015033 (D) 2016532 (E) 2017033